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Book reviews

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Approximate Approximations. *By Vladimir Maz'ya and Gunther Schmidt.* Mathematical Surveys and Monographs, Volume 141, AMS, Providence, RI, 2007. xiv + 349 pp., hardcover. USD 89, ISBN 978-0-8218-4203-4.

Most standard numerical approximation methods are designed to be *convergent*, i.e., the approximation error goes to zero as the method is refined. The main – and fundamentally new – idea driving the concept of *approximate approximations* is to work with a theoretically non-converging method which can be tuned by choosing a scaling parameter appropriately so that it does converge for all practical purposes.

One can not illustrate this better than with an example lifted straight from the first chapter of the book. The authors show on page 3 using a simple argument based on the use of Fourier series and the Poisson summation formula that the function

$$\theta(x, \mathcal{D}) = \frac{1}{\sqrt{\pi \mathcal{D}}} \sum_{m=-\infty}^{\infty} \mathrm{e}^{-(x-m)^2/\mathcal{D}} \tag{1}$$

with scaling parameter $\mathcal{D}$ is equal to

$$\theta(x, \mathcal{D}) = 1 + 2 \sum_{\nu=1}^{\infty} \mathrm{e}^{-\pi^2 \mathcal{D} \nu^2} \cos 2\pi \nu x. \tag{2}$$

Using this simple example we can get a good feel for what approximate approximations are all about. Equation (1) represents a quasi-interpolant with Gaussian generating functions for the data

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sampled at the integers, m, from the constant function $u(x) = 1$. And equation (2) shows that the quasi-interpolant indeed recovers the original function — except for the *saturation error*

$$2\sum_{\nu=1}^{\infty} \mathrm{e}^{-\pi^2\mathcal{D}\nu^2} \cos 2\pi\nu x.$$

This saturation term is oscillatory, and for $\mathcal{D} \geq 1$ its modulus can be bounded by $1.04 \cdot 10^{-4\mathcal{D}}$. Consequently, we can use the scaling parameter $\mathcal{D}$ to ensure that the quasi-interpolant (1) is *exact to within any desired accuracy*. For example $\mathcal{D} = 4$ will yield double-precision floating point accuracy.

All of this has a multivariate analog — in particular with radial basis functions. Thus, while a quasi-interpolant with Gaussians does not reproduce constants, it can do so for all practical purposes provided the scale parameter $\mathcal{D}$ is chosen large enough. Not being able to reproduce constants (or other low-degree polynomials) was long believed to be a flaw of radial basis functions. The theory of approximate approximations provides an elegant way to show that one does not need to require exact polynomial reproduction (or partition of unity and vanishing moments) in order to have a practical numerical method. All one needs to ensure is that these properties are almost satisfied.

The book begins with an "Exercise for a freshman" (the same example I picked above to illustrate the idea of approximate approximations). While this is a great example presented in a very nice way, the fact that one needs to know about Fourier series and the Poisson summation formula clearly shows that this is not for a first-year college student — not even for the usual freshman graduate student. Having calibrated our expectations/prerequisites, the first chapter of *Approximate Approximations* is an excellent introduction and motivation for the reader to delve into a more detailed study of the subject in later chapters. If you are not yet familiar with the concept of approximate approximations then you should definitely give the first chapter of this book a try!

As just indicated, the book will probably not be an easy read for many members of its target audience ("graduate students and researchers interested in applied approximation theory and numerical methods for solving problems of mathematical physics"). However, it is very well organized, the notation used is consistent throughout, and all derivations are clear and detailed. The new ideas that arise in this theory of approximate approximations (perhaps they are even fundamental philosophical concepts) are compared to those prevalent in the existing literature on approximation theory and numerical analysis. This book is definitely much more than a mere collection of papers by its two authors (as well as several other co-authors).

The pointwise error estimate for approximation of more general smooth univariate functions u by Gaussian quasi-interpolation of the form

$$\mathcal{M}_{h,\mathcal{D}}u(x) = \frac{1}{\sqrt{\pi\mathcal{D}}} \sum_{m=-\infty}^{\infty} u(mh)\mathrm{e}^{-(x-mh)^2/\mathcal{D}h^2}$$

is of the order $\mathcal{O}(\mathcal{D}h^2)+\varepsilon(\mathcal{D})$ (also shown in Chapter 1). As before, the saturation error $\varepsilon(\mathcal{D})$ can be made arbitrarily small by choosing $\mathcal{D}$ large enough. This results, for a fixed scale parameter $\mathcal{D}$, in a convergent (up to any desired accuracy) stationary approximation scheme based on Gaussians.

So far, we have only mentioned the approximate partition of unity property of Gaussians. In order to obtain basis functions that provide higher approximate approximation orders one multiplies the Gaussians by appropriately chosen generalized Laguerre polynomials (which can also be expressed as Hermite polynomials in the univariate case). This yields a basis with any desired number of approximately zero moments. Section 1.2.4 contains already some examples of this construction. Any other approximate partition of unity can be used to generate similar families of approximate approximants. Several such families are among the material discussed in detail in Chapter 3 of the book.

In addition to the excellent motivational Chapter 1 and Chapter 3 on the construction of approximate higher-order generating functions, there are eleven other chapters. Chapter 2 contains a detailed discussion of error estimates for the quasi-interpolants used here. This includes pointwise and L_p estimates of function values as well as derivatives both of a global and local nature. The latter are important for the discussion of approximation problems on bounded domains and approximate wavelets.

After presenting the fundamental theory underlying the concept of approximate function approximation by quasi-interpolation the authors devote a large portion of the book to applications of these ideas. One of the main motivations driving the development of the theory starting in the early 1990s was the fact that one can use the special quasi-interpolants provided by the approximate approximation theory to obtain semi-analytic cubature formulas which in turn can be used to solve integral equations involving singular kernels. Chapter 4 provides the main error estimates for the approximation of these integral operators. In this chapter we also find (Theorem 4.6) an error estimate for approximate approximation in Sobolev spaces of negative order. Since the saturation error of approximate approximations is of an oscillatory nature one actually ends up with *convergent* methods if measured in these weaker norms! Chapters 5 and 6 contain a detailed discussion of the cubature of special potentials from mathematical physics including diffraction potentials, potentials of advection-diffusion operators, and elastic and hydrodynamic potentials.

The book also contains a section, Section 7.3, on approximate interpolation with Gaussians. Here, the authors derive an approximate cardinal basis for the space spanned by Gaussians (on all of $\mathbb{R}^n$) given by the tensor product

$$\Psi_{\mathcal{D}}(\mathbf{x}) = \frac{1}{(\pi\mathcal{D})^n} \prod_{j=1}^{n} \frac{\sin \pi x_j}{\sinh \frac{x_j}{\mathcal{D}}}, \quad \mathbf{x} = (x_1, \ldots, x_n).$$

With such a representation one can of course completely avoid the solution of linear systems as they arise in the usual discussion of multivariate interpolation with Gaussian radial basis functions. The authors also show that the approximation order of this method is approximately spectral.

While the theory of approximate approximations in its basic form lives on the entire Euclidean space $\mathbb{R}^n$, there are also parts of the book that discuss what happens if we consider problems on more general grids, at scattered data sites, on manifolds or on bounded domains. These topics are covered in Chapters 9–11.

One of the tools needed to deal with the refinements required for boundary layers of domains is the theory of approximate wavelets. This topic is presented in Chapter 8. There the usual refinement equation is replaced by an approximate refinement equation of the form

$$\eta\left(\frac{\mathbf{x}}{\sqrt{\mathcal{D}}}\right) = \mathcal{D}^{-n/2} \sum_{\mathbf{m}\in\mathbb{Z}^n} \tilde{\eta}\left(\frac{\mu\mathbf{m}}{\sqrt{\mathcal{D}}}\right) \eta\left(\frac{\mathbf{x}-\mu\mathbf{m}}{\mu\sqrt{\mathcal{D}}}\right) + \text{small remainder},$$

where for the Gaussian

$$\eta(\mathbf{x}) = \mathrm{e}^{-|\mathbf{x}|^2}$$

one has the mask function

$$\tilde{\eta}(\mathbf{x}) = \frac{\mathrm{e}^{-|\mathbf{x}|^2/(1-\mu^2)}}{(\pi(1-\mu^2))^{n/2}}.$$

Note that in this theory the refinement parameter $\mu < 1$ is not limited to the standard choice $\mu = \frac{1}{2}$. As in the basic approximate approximation setting, the actual size of the small remainder term depends on the scale $\mathcal{D}$ of the Gaussian.

The final two chapters of the book are concerned with applications of the approximate approximation idea to the numerical solution of certain linear (Chapter 12) and nonlinear (Chapter 13) PDEs including numerical examples. Most of these examples — except for the discussion of boundary integral equations in Section 12.2 — live on all of $\mathbb{R}^n$.

The book actually has a good deal of numerical examples sprinkled throughout. I consider this is a great and valuable addition to an otherwise very technical book. However, I found that there always seemed to be a little bit of information missing, so that one is not always able to reproduce the examples without some guesswork about the missing details/parameter values. This begins right with the first figures on page 1 where we see graphs on the interval $[-4, 4]$ when the function under consideration is defined and approximated on the entire real line. Clearly, we can not perform the numerical computations on the entire real line. But we can not just do them on $[-4, 4]$ either since in that case one would observe a boundary effect. Of course, this detail may seem irrelevant (perhaps even distracting at this point), but without having this information the reader may have a hard time reproducing the graphs.

Finally, the index of the book is useless. In fact, one is much more likely to succeed identifying a particular section of interest by searching through the rather detailed table of contents at the front of the book than by consulting the skimpy little one-page index on page 349.

In summary I recommend this book very highly to anyone interested in solving function approximation problems or problems of the integral equation type mentioned above. The material is not that easy to digest, but the idea of using a numerical method that converges only within a certain range of practical interest is a new and exciting one that I have not yet seen enough of in the literature.

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Limit theorems of polynomial approximation with exponential weights. *By Michael I. Ganzburg*. Memoirs of the American Mathematical Society, Volume 192, AMS, Providence, RI, 2008. 161 pp., softcover. USD 69, ISBN 978-0-8218-4063-4.

This monograph is structured around limit relations between the errors of polynomial approximation in weighted metrics. A classical limit relation is due to Bernstein. Let $\mathcal{P}_n$ be the class of algebraic polynomials of degree $\leq n$ and B_σ the class of all entire functions of exponential type $\sigma > 0$. Denote

$$E_n(f, L_p(\Omega)) := \inf_{P\in\mathcal{P}_n} \|f - P\|_{L_p(\Omega)}, \qquad A_\sigma(f, L_p) := \inf_{g\in B_\sigma} \|f - g\|_{L_p(\mathbb{R})}.$$

Bernstein showed that

$$\lim_{n\to\infty} E_n(f, L_\infty[-n/\sigma, n/\sigma]) = A_\sigma(f, L_\infty) \tag{1}$$

for continuous f of polynomial growth on $\mathbb{R}$ and for almost all $\sigma > 0$. The author of the monograph proved in 1991 that “almost all” can be replaced by “all” if we squeeze the interval by a factor of order $1 - n^{-\delta}$ for some $\delta \in (0, 2/3)$.

Similar limit relations hold in the L_p norm, $p > 0$:

$$\lim_{n\to\infty} n^{1/p} E_n(f((n/\sigma)\cdot), L_p[-1, 1]) = \sigma^{1/\sigma} A_\sigma(f, L_p). \tag{2}$$

Again, this fact is a combination of contributions of Bernstein and Ganzburg.

In this monograph, weighted analogues of (1)–(2) are discussed. The main tool are estimates of $E_n(f, L_{p,W})$ in the norm

$$\|f\|_{L_{p,W}(I)} := \left(\int_I (|f(x)|\, W(x))^p \,\mathrm{d}x \right)^{1/p}.$$

The weights $W(x) = \exp(-Q(x))$ on $\mathbb{R}$ are taken from the exponential class $\mathcal{F}(C^2)$, introduced by Levin and Lubinsky (see the monograph of A. L. Levin D. S. Lubinsky, “Orthogonal Polynomials for Exponential Weights”, Springer, New York, 2001). In this situation the scaling factor n in (1) and (2) has to be replaced by the value $b_n \approx n/a_n$, where a_n is the Mhaskar-Rakhmanov-Saff number (endpoint of the equilibrium measure). This value appears also in the estimate

$$|P(z)| \le \mathrm{e}^{b_n|z|}\, \|PW\|_{L_\infty(-c,c)}, \qquad P \in \mathcal{P}_n, z \in \mathbb{C},$$

proved in this monograph.

Here is a brief overview of the content of the book. Chapter 1 is a nicely written introduction with historical background and a concise explanation of the main contributions of this work. Chapter 2 contains the statement of the central results of the monograph, both the limit relations and the weighted inequalities. Chapters 3–6 provide the necessary ingredients for the proofs (gathered in Chapter 7): properties of harmonic functions, polynomial inequalities and entire functions.

The second part of the book contains applications of the limit relations in approximation theory. Chapter 8 is devoted to approximation of individual functions and to best constants. A famous contribution of Bernstein is the study of $E_n(f_\lambda, L_p[-1, 1])$, where $f_\lambda(x) = |x|^\lambda$. The asymptotic relation

$$\lim_{n\to\infty} n^{\lambda+1/p} E_n(f_\lambda, L_p[-1, 1]) = B_{\lambda,p} = A_1(f_\lambda, L_p) \tag{3}$$

is a direct consequence of the limit relations. Ganzburg shows that $B_{\lambda,p} < \infty$ in (3) for $p \in (0, \infty]$ and $\lambda > \max(-1, -1/p)$. Further extensions of (3) for λ-homogeneous functions (eventually times $\log^k |x|$) are proved as well. There are also results on convergence of polynomials, of the form

$$\lim_{k\to\infty} b_{n_k}^{-1} P_{n_k}(z/b_{n_k}) = g_0(z) \tag{4}$$

where $\{P_n\}$ are uniformly bounded (in $L_{p,W}$) polynomials. A more precise version of (4) is obtained assuming that P_n is the n-th orthogonal polynomial corresponding to $W \in \mathcal{F}(C^2)$ (a Mehler-Heine formula).

Chapter 9 contains multidimensional versions of limit theorems of polynomial approximation with exponential weights.

Finally, Chapter 10 gathers a miscellany of refinements of results on polynomial approximation with special exponential weights, mostly with the canonical weights

$$W_\alpha(x) := \exp(-|x|^\alpha), \quad \alpha > 0.$$

In particular, the boundary weight W_1 is considered in Section 10.2. It is known that polynomials are dense in the L_{∞,W_1} norm in the class of continuous functions on $\mathbb{R}$ such that $\lim_{|x|\to\infty} |f(x)|W_1(x) = 0$. However, $W_1 \notin \mathcal{F}(C^2)$ and $b_n = \infty$, so many technical results have to be reproved. In this case, limit relation of the type (2) is not obtained; instead, the author finds some two-sided inequalities. Finally, the very interesting indeterminate case W_α, $0 < \alpha < 1$, is analyzed in Section 10.3; further progress is made in the study of the closure of all polynomials in L_{∞,W_α}, whose complete description is still an open problem.

The books ends with an Appendix containing some technical material, a list of bibliography with 122 items, and a very short Index.

In summary, this book is suitable for approximators, particularly for those interested in weighted polynomial approximation or orthogonal polynomials. It is well organized and not difficult to read, especially the first part, and gathers all the important tools, results and properties, as well as detailed proofs and references.

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Affine Densities in Wavelet Analysis. *By Gitta Kutyniok.* Lecture Notes in Mathematics, Volume 1914, Springer Verlag, Berlin, 2007. xii + 142 pp., softcover. 30€, ISBN 978-3-540-72916-7.

Let $\mathbb{A} = \mathbb{R}^+ \times \mathbb{R}$ denote the affine group. Given a finite collection $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})$, time-scale sequences $\Lambda_1, \ldots, \Lambda_L \subseteq \mathbb{A}$, and weight functions $w_\ell : \Lambda_\ell \to \mathbb{R}^+$, $\ell = 1, \ldots, L$, we can consider the associated weighted (irregular) wavelet system defined by

$$\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell, w_\ell) = \bigcup_{\ell=1}^{L} \left\{ w_\ell(a,b)^{1/2} \frac{1}{\sqrt{a}} \psi_\ell\left(\frac{x}{a} - b\right) \right\}_{(a,b)\in\Lambda_\ell}.$$

The fundamental problem considered in the monograph under review is to obtain necessary (and a few sufficient) conditions on the structure of the time-scale sequences $\Lambda_1, \ldots, \Lambda_L \subseteq \mathbb{A}$, and on the weights $w_\ell : \Lambda_\ell \to \mathbb{R}^+$, such that an associated wavelet system forming a frame for $L^2(\mathbb{R})$ exists. The existence of a wavelet frame imposes some geometrical restrictions on the time-frequency sequences $\{\Lambda_\ell\}$, the intuitive idea being that the points in Λ_ℓ cannot be “too sparse” nor can they accumulate “too densely” in any suitable region of $\mathbb{A}$. To quantify the geometric constraints, the author introduces suitable upper and lower weighted affine densities on the affine group $\mathbb{A}$ that take into account the structure of $\mathbb{A}$. In terms of the weighted density, the necessary conditions simply say that the time-frequency sequences must have positive lower density and a finite upper density.

The book is the author’s Habilitationsschrift in mathematics at the Justus-Liebig-Universität Gießen. I found the monograph to be very readable, and it should prove valuable for anyone wishing to learn more about the finer properties of wavelet frames.

Content of the book. The book is divided into seven chapters that can be grouped into four parts.

Chapters 1 *and* 2: Contains some background on basic frame theory. Wavelet and Gabor frames are introduced.

Chapters 3, 4, *and* 5: Introduces the notion of weighted affine density. Qualitative and quantitative density conditions are considered. The author derives necessary conditions on the upper and lower weighted affine density for the existence of a weighted irregular wavelet frame with a finite number of generators. A highlight of Chapter 5 is Theorem 5.6 that establishes a relationship between the affine weighted density, the frame bounds, and the admissibility constants for the analyzing wavelets.

Chapter 6: Several results on the so-called Homogeneous Approximation Property for wavelet frames are derived.

Chapter 7: Density results for Gabor systems and associated shift-invariant systems are considered.

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Polynomial Convexity. *By Edgar Lee Stout*. Progress in Mathematics, Volume 261, Birkhäuser Boston, 2007. x + 439 pp., hardcover. 70€, ISBN 978-0817645373.

Polynomial convexity is a phenomenon of function theory in several complex variables, closely related to approximation theory, analytic continuation, etc. Polynomially convex hull $\hat{K}$ of a compact set $K \subset \mathbb{C}^N$ is the set of points where the maximum principle for holomorphic polynomials is valid:

$$\hat{K} = \left\{ z \in \mathbb{C}^N : |p(z)| \leq \max_K |p|, \text{ for every polynomial } p \right\}.$$

Thus, if a sequence of polynomials in $z_1, \ldots, z_N$ is uniformly converging on K, then it is uniformly converging on $\hat{K}$ also, with the same rate.

For $N = 1$ the polynomially convex hull is rather simple: $\hat{K}$ is the union of K and all bounded components of $\mathbb{C} \setminus K$. But for $N > 1$ the situation becomes much more complicated and interesting. Say, if K contains the boundary of a bounded complex analytic subvariety $V \subset \mathbb{C}^N \setminus K$, then $V \subset \hat{K}$ (an analytic structure in the hull), but in general $\hat{K} \setminus K$ can be rather complicated, containing no holomorphic disc.

The set K is called polynomially convex if $\hat{K} = K$. The study of polynomial convexity started about 50 years ago, and all these years the author of the book was one of the most active and prominent mathematicians working in this direction. It is fortunate that Stout he has written a book condensing the main achievements in the area from the very beginning till the very last years.

The monograph contains a short preface and eight chapters.

Chapter 1 is an introduction to the subject. It contains the discussion of the notions of polynomial and rational hulls, with the stress on analytic structures in the hulls, with many interesting, unexpected and amusing examples. There are short, condensed introductions in uniform algebras and plurisubharmonic functions, and detailed study of Cauchy–Fantappiè integral which is used in the proof of the Oka–Weil theorem, building a bridge between analytic approximation theory in several complex variables and geometrical theory of polynomial convexity.

Chapter 2 on general properties of polynomially convex sets shows the vast collection of tools and methods used in the analysis of polynomial convexity: different applications of the 1st and 2nd Cousin problems, characterization of polynomially convex sets in terms of positive currents (theorem of Duval and Sibony) and holomorphic discs (Poletsky theorem), numerous applications of differential topology (Morse theory) and algebraic topology to topological properties of polynomially convex sets in $\mathbb{C}^N$, and generalizations of Stein manifolds. The exposition contains complete proofs and clear explanations of used notions and results from other areas of mathematics. There is a rich variety of impressive, and I confess that several of them were totally new for me. Together, Chapters 1 and 2 can be recommended as a nice graduate level course in Complex Analysis in several variables.

Chapters 3 and 4 are devoted to fairly self-contained discussion of polynomially convex hulls of compact sets of finite length in $\mathbb{C}^N$ (the connected case is studied in Chapter 3, and the more general situation, in Chapter 4). The exposition is based mainly on pioneering works of H. Alexander. The main result of Chapter 3 states that the complementary polynomial hull of a rectifiable simple closed curve Γ in $\mathbb{C}^N$, if not empty, is a purely one-dimensional complex-analytic subvariety of $\mathbb{C}^N \setminus \Gamma$ with finite area (the result is rather nontrivial even for smooth Γ). It is proved in Ch. 4 that the same is true for so-called geometrically 1-rectifiable sets Γ. For that, one needs delicate geometrical properties of plane sets of finite length, solid portion of geometric measure theory, theory of currents, one-dimensional varieties, further results on plurisubharmonic functions, etcetera. The majority of auxiliary results (including Besicovitch theorems) are given with complete proofs. The results are applied to the problems of analytic continuation (removable singularities) of complex one-dimensional varieties. Chapters 3 and 4, with short introduction from Chapter 1, can also be recommended as a course for graduate students interested in Complex Analysis and geometric measure theory.

Chapter 5 contains further results on polynomial convexity: isoperimetric inequalities for hulls, theorems on removable singularities and analytic continuation of CR-functions and holomorphic functions, the study of hulls of real two-dimensional surfaces in pseudoconvex boundaries and surfaces with isolated complex tangents.

Chapter 6 is the most interesting for specialists in approximation theory. It is devoted to the uniform polynomial and holomorphic approximation of continuous functions on totally real sets in $\mathbb{C}^N$. After preliminaries on totally real sets and the discussion of holomorphically convex sets, a theorem on uniform holomorphic approximation on the unions of holomorphically convex and totally real sets follows. Next, elements of rational approximations on plane compact sets are given, along with a series of results on subalgebras of continuous functions and uniform holomorphic approximation on two-dimensional surfaces and some real-analytic subsets of $\mathbb{C}^N$. The chapter is finished by the results on tangential approximation by entire functions on some unbounded closed subsets of $\mathbb{C}^N$.

Chapter 7 is devoted to the study of boundary properties of one-dimensional subvarieties of strictly pseudoconvex domains. It starts with description of so-called peak-interpolation sets on the boundary of such a domain D and conditions of corresponding convexity of rectifiable curves on ∂D. Then the boundary regularity of a one-dimensional variety $V \subset D$ with regular $\partial V \subset \partial D$ is studied and boundary uniqueness theorem for such V is proved. The results are applied to the proof of a theorem of M. Lawrence: if X is a compact set of finite length in C^2-boundary of a strictly convex domain in $\mathbb{C}^N$ then $\hat{X} \setminus X$ is analytic subvariety of $\mathbb{C}^N \setminus X$.

Examples and counterexamples on polynomial convexity and related topics are collected in final the Chapter 8, with statements and proofs. The following topics are considered: projective capacity in $\mathbb{P}^N$ and its relation with hulls of circular sets in $\mathbb{C}^{N+1}$, polynomial hulls of compact subsets of unions of totally real subspaces of $\mathbb{C}^N$, polynomial convexity of the union of disjoint balls, holomorphicity of pluripolar graphs, the hulls of deformations of polynomially convex sets and sets with various symmetries.

I can note only one subject on polynomial convexity worthwhile to include in the book, namely, the description of polynomially convex hulls of multifunctions over the unit circle (see

[1,2]). It needs some extended preliminaries on quasiconformal mappings, Riemann–Hilbert problem, etc., and could enlarge the volume substantially, but maybe it is a subject for a following edition.

The book of Lee Stout demonstrates brightly, how the purely analytical maximum principle involves in its orbit many other branches of mathematics, looking often very distant from Analysis or Approximations. The book is written by a master of the subject. It is rather saturated and condensed, it is a book for slow reading. For specialists in several complex variables it will be a nice handbook and source of reference. It is highly recommended also for graduate students.

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Gabor and Wavelet Frames. *By Say Song Goh, Amos Ron, and Zuowei Shen (editors).* Lecture Notes Series, Vol. 10, Institute for Mathematical Sciences, National University of Singapore, World Scientific, Singapore, 2007. xii + 214 pp., hardcover. USD 75, ISBN 978-981-270-907-3.

Data representation is a big deal in today's world. The size and complexity of modern data sets place a premium on finding effective tools for representing signals. Wavelets and Gabor systems are two especially successful approaches to signal representation that have risen to prominence in the past two decades. In a nutshell, a wavelet system $\{2^{m/2}\psi(2^m t - n) : m, n \in \mathbb{Z}\}$ uses translated and dilated versions of a window function ψ to give *time-scale* decompositions, whereas a Gabor system $\{e^{2\pi i amt} g(t - bn) : m, n \in \mathbb{Z}\}$ (with fixed $a, b \neq 0$) uses translated and modulated versions of a window function g to give *time-frequency* decompositions.

This edited volume on wavelet and Gabor frames grew out of the thematic program "Mathematics and Computation in Imaging Science and Information Processing" that was held in the Institute for Mathematical Sciences at the National University of Singapore in 2003 and 2004. The volume contains five chapters written by leading experts on wavelets and Gabor systems, and provides expositions and research tutorials on theoretical aspects of the field. The individual chapters are well written and the volume nicely conveys a tutorial tone.

The volume is implicitly divided into two parts, with the first three chapters covering Gabor analysis and the final two chapters focusing on wavelet theory. The volume begins with a brief descriptive preface by the editors. *Chapter* 1, "A Guided Tour from Linear Algebra to the Foundations of Gabor Analysis", by Hans Feichtinger, Franz Luef, and Tobias Werther, broadly surveys the vast landscape of time-frequency analysis, including important topics such as frames, Riesz bases, time-frequency representations, the spreading function, the Feichtinger algebra, the Balian-Low theorem, Wilson bases, and density theorems. *Chapter* 2, "Some Iterative Algorithms to Compute Canonical Windows for Gabor Frames", by A.J.E.M. Janssen, is a focused study of five iterative algorithms that are used to compute canonical dual frames and canonical tight frames associated with Gabor frames. Careful attention is given to convergence rates and the role of frame constants. *Chapter* 3, "Gabor Analysis, Noncommutative Tori and Feichtinger's Algebra", by Franz Luef, establishes several interesting connections between

Gabor analysis and operator algebras, in particular Morita equivalence of noncommutative tori. The chapter provides new insights into the Wexler-Raz theorem and the structure of the Feichtinger algebra. *Chapter* 4, “Unitary Matrix Functions, Wavelet Algorithms, and Structural Properties of Wavelets”, by Palle Jorgensen, gives an introduction to wavelet subband filtering that highlights key topics such as multiresolution analysis, wavelet packets, lifting algorithms, and factorization theorems for matrix functions. *Chapter* 5, “Unitary Systems, Wavelet Sets, and Operator-Theoretic Interpolation of Wavelets and Frames”, by David Larson, discusses the role of operator theory in wavelet analysis, and crisply explains important ingredients such as the local commutant of a wavelet system and interpolation pairs of wandering vectors. Work on minimally supported frequency wavelets and their wavelet sets is also presented as a part of this theory.

The chapters in “Gabor and wavelet frames” successfully draw connections with different areas of mathematics, notably with operator theory and operator algebras in Chapters 3, 4, and 5. This leads to novel treatments that differ from some of the classical approaches. The overall volume is likely to appeal to researchers with more theoretical interests, but it contains relevant material for both pure and applied mathematicians. The same Lecture Notes Series also contains a more applied companion volume [6] by the same editors.

This book is not intended (and not suitable) as a textbook on either wavelets or Gabor systems, but the diverse assortment of chapters can serve as a valuable source of introductory and advanced material for special topics courses. The individual chapters contain sketches and complete proofs of many theorems, and where proofs are not given it is done so with expository aims in mind. This book will probably appeal most to those who already have some knowledge of the field. For those in search of a textbook on wavelets or Gabor systems, the classics [3, 7, 8] are outstanding places to start. As an edited volume, this book successfully conveys a tutorial tone and fits well in the Lecture Notes Series. For comparison, it contains a smaller assortment of chapters than similar edited volumes, e.g., [1, 2, 4, 5], and lacks an index. The book is remarkably free of typographical errors and omissions (the back cover contains a rare exception).

The editors have put together a fine volume on wavelets and Gabor analysis, and the contributing authors have presented informative and well written chapters.

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Handbook of Approximation Algorithms and Metaheuristics. *By Teofilo F. Gonzalez (editor)*. CRC Computer and Information Science Series, Volume 13, Chapman & Hall, 2007. 1432 pp., hardcover. USD 130, ISBN 978-1584885504.

Approximation algorithms for solving computationally intensive optimization problems have been widely studied in the literature for the last forty years. The book provides a comprehensive and thorough exposition of the different methodologies used so far to approach this problem, including the so-called *Metaheuristics*, which have been successfully applied in the last two decades.

Regarding its contents, most of the approximation algorithms available for optimization problems are presented. It covers different areas, as *linear programming*, *probabilistic analysis*, *randomized algorithms*, including methods like *neural networks*, *greedy search*, *simulated annealing*, *ant colony optimization* and *tabu search*, to mention a few.

These algorithms have been applied in different fields, such as computer science, operations research, computer engineering, applied mathematics, bioinformatics, geography, economy, etcetera, which makes this contribution relevant to a wide audience.

The book is organized in 6 parts, and 86 chapters. Chapters are not very lengthy, 8–15 pages each, and each one explains a different algorithm, its main ideas, goals and properties at an intermediate level. The exposition of the algorithms is not restricted merely to pseudocodes, but the main properties of the method are discussed and proven, and its performance explained in detail. Every chapter ends with its own bibliography, most of them with a great number of references, so that it becomes extremely easy to gather more information about each specific algorithm.

The first 3 parts of the book explain the algorithms in a general framework; in Part I, basic methodologies are presented; local search, neural networks and metaheuristics are shown in Part II, while Part III is devoted to multiobjective optimization, sensitivity analysis and stability.

The last 3 parts comprise practical applications of the former algorithms. Part IV deals with traditional applications such as best packing and location or assignment problems. Part V contains computational geometry and graph applications, triangulations, connectivity, partitioning or search problems. And the last one focuses on large-scale and emerging applications related to very diverse problems such as scheduling on wireless channels, microarray analysis or economic applications.

Besides its content, we liked also the way the topics are written. According to its title, it is a handbook, but it goes beyond that: it is not a simple collection of short surveys (written even by different authors), but the methods are presented in such a way that different units are linked in a coherent form, similar to a monograph rather than a handbook.

To summarize, this book may become a reference, since it covers a gap in the existing literature, where it is difficult to find a single book in which most of the current standard methods are presented. It is extremely useful, not only for researchers in the field of approximation algorithms, but also for practitioners, or even for educators at a graduate level, as a guide in the lecture room, even though this book does not include exercises or examples.

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Bounded Analytic Functions. *By John B. Garnett.* Graduate Texts in Mathematics, Volume 236, Springer Verlag, Berlin, revised edition, 2007. 466 pp., hardcover. 55€, ISBN 978-0-387-33621-3.

This is a revised first edition of a classical book written by a major researcher in Harmonic and Complex analysis. The first edition appeared in 1981 and has had a great impact in the area. The present one is the same as the first except for a few corrections. The book is beautifully written and has been used both as textbook and a primary reference for researchers around the world. It was awarded with the 2003 AMS Steele Prize for Exposition, from which we quote

...The book, which contains a wide range of beautiful topics in analysis, is extremely well organized and well written, with elegant, detailed proofs. The book has educated a whole generation of mathematicians with backgrounds in complex analysis and function algebras. It has had a great impact on the early careers of many leading analysts and has been widely adopted as a textbook for graduate courses and learning seminars ...

The book is concerned with the theory of Hardy spaces of analytic functions in the unit disc of the complex plane. This is placed in an area where Complex and Harmonic analysis interact. Instead of being an exhaustive account, the author chooses several significant problems in the area and explored them at a considerable depth. Examples in this direction are Fefferman's H^1 − BMO duality, Carleson's Corona Theorem and the description of Douglas algebras. Other topics as the interaction of H^∞ and operator theory have been minimized.

Real variable methods and stopping time constructions play a central role in the book. Maximal functions, Littlewood-Payley integrals, Carleson measures, good λ inequalities are systematically used through the text. Many of these techniques extend naturally to the Euclidean space setting and beyond, but the author has limited the discussion to the one dimensional case. It is worth mentioning that in several occasions, the presentation in the text is very original and differs considerably from the original sources.

In perspective, both the choice of the topics and the techniques seem to be the correct point of view. Moreover, the book seems to be especially valuable for those interested in the theory in several real or complex variables and the presentation is coherent with the program of extending the classical Hardy space theory to Euclidean spaces. The text is selfcontained but demands a lot of attention from the reader.

The book is concerned with the theory of analytic functions in the unit disk and the upper half-plane. The most classical topics such as the Hardy–Littlewood maximal function, the inner-outer factorization in Hardy spaces, the description of the invariant subspaces of the Hardy space H^2 and the conjugate function, are presented in the first three chapters. These results constitute some of the first relevant results in the area and were due to P. Fatou, G. Hardy, J. Littlewood, the Riesz brothers and A. Beurling. The more recent result of D. Burkholder, R. Gundy and M. Silverstein on the description of Hardy classes in terms of non-tangential maximal functions are also included. It is worth mentioning that the author adopts a real variable point of view with an extensive use of Maximal functions, stopping time arguments and good λ inequalities. Chapter IV is devoted to several extremal problems, among which we mention the problem of best approximation in H^∞ and the classical Nevanlinna-Pick interpolation problem. Here the author presents a very original approach to an important result of V. Adamyan, D. Arov and M. Krein which is very different from their spectral theory approach. Next Chapter develops the background from uniform algebras which is needed for the deep analysis of the algebra H^∞ which is presented in the next chapters. Chapter VI is devoted to a careful study of the space BMO of functions of bounded mean oscillation in the unit circle or the real line.

C. Fefferman's duality theorem which identifies BMO as the dual of the real Hardy Space H^1, the John-Nirenberg theorem as well as the relation of BMO functions with Carleson measures and the B. Muckenhoupt A_p weights are presented. Chapters VII and VIII are devoted to two deep results by L. Carleson in the sixties which modeled the whole area: the characterization of interpolating sequences and the Corona Theorem. BMO functions, interpolating sequences and the Corona Theorem are linked through the notion of Carleson measure. The Corona Theorem states that the unit disk is dense in the maximal ideal space of H^∞. This has a very concrete formulation: if $f_l, \dots, f_n$ are functions in H^∞ which are not simultaneously small, then there exist $g_1, \dots, g_n$ in H^∞ such that $f_1 g_1 + \dots + f_n g_n = 1$. The heart of Carleson's proof of the Corona Theorem is a complicated construction which produces, for a given function f in H^∞, a system Γ of rectifiable closed curves which act as level sets of $|f|$ and such that arc-length on Γ is a Carleson measure. This is beautifully presented in the book with the use of stopping time arguments, subharmonicity and Maximal Functions. This type of constructions are now known as Corona Constructions and have been developed in several contexts in Harmonic Analysis and Geometric Measure Theory. T. Wolff's proof of the Corona Theorem is also presented in Chapter VIII. Next Chapter is devoted to Douglas algebras, which are the closed algebras between H^∞ and L^∞, and contains the deep results by S.Y. Chang and D. Marshall on the description and structure of these algebras. It is worth mentioning that in this chapter some of the results from many of the earlier parts of the book are used. Finally, last chapter is devoted to the results by K. Hoffman which relate interpolating sequences with the structure of the maximal ideal space of H^∞ and closes with a discussion of an interpolating problem by bounded harmonic functions by L. Carleson and the author and with an approximation result by P. Jones.

Each chapter contains a section called *Notes* where brief historical comments and the references of the original sources are given. Each Chapter ends with a section called *Exercises and Further Results*. These add much more to the theory presented in the book and many of them are accompanied by detailed hints and references.

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Analysis and Probability; Wavelets, Signals, Fractals. *By P. E. T. Jorgensen.* Graduate Texts in Mathematics, Volume 234, Springer Verlag, Berlin, 2006. xlvii + 276 pp., hardcover. 43€, ISBN 978-0-387-29519-0.

Wavelet theory is a relatively new branch of mathematics, even if some fundamental ideas can be traced back about ninety years. And some of the ideas have been around in signal and image processing for some time. The concept of *wavelets* were first introduced (in French) by Morlet and Grossman in 1982. During the first few year, a notable contributions to wavelet theory was made by Goupillaud, Grossman and Morlet's and Daubechies, who in 1988 gave a systematic construction of orthogonal wavelets with compact support. A huge step forward was then the introduction of multiresolution, or nested subspaces, by Mallat in 1989. But textbooks were slow to catch up. As I taught my first class on wavelets about ten years ago, there were hardly any textbook out there to use, with exception of [4, 9, 10] and those are not really textbooks for a class. Now, almost ten years later, the situation is different and there is no shortage of books, stressing different aspect of wavelet theory. So, why one more book?

Looking through the book by Palle Jorgensen, this is the wrong question. This is not the typical book on wavelets. The book is based on courses that Palle has taught at the University of Iowa for several years. It is also based on parts of his–and coworkers–most recent research.

The central topic is the many faces of iterations and their fixed points. The first thing that comes to mind is fractal geometry. But the same philosophy is the basic behind the construction of wavelets and scaling functions which are fixed points under iterations

$$\varphi(t) = N \sum_{k\in\mathbb{Z}} a_k \varphi(Nt - k). \tag{1}$$

But there is no a priory reason for (1) to have a solution in $L^2(\mathbb{R})$! This brings up fractal constructions and spectral sets. As special example of a wavelet are the minimal wavelets or those that corresponds under the Fourier transform to the indicator function of a measurable subset Ω in $\mathbb{R}^d$, $|\Omega| < \infty$. A simple example is $\Omega = [-1, -1/2) \cup (1/2, 1]$ which tiles the line under dilation by $\{2^j\}_{j\in\mathbb{Z}}$ and translation by integers, connecting the wavelet theory to the beautiful work of Fuglede [5].

The sets Ω that give rise to wavelets are – not surprisingly – called *wavelet sets*. But, those sets are in particular, *spectral sets*, i.e., there exists a set $\Gamma \subset \mathbb{R}^d$ such that $\{e_\gamma\}_{\gamma\in\Gamma}$, $e_\gamma(x) = \mathrm{e}^{2\pi \mathrm{i} x\cdot\gamma}$, forms an orthogonal basis for $L^2(\Omega)$. This question is just as meaningful for fractal sets, which is the starting point for the joint work of Jorgensen and Pedersen [6, 7, 8]. This is partially discussed in Chapter 4 in the book under review.

But this is a slight jump forwards. The book starts with an introduction to probability, random walks, and Ruelle operators. Let $(X, \mathcal{B}, \mu)$ be a measure space and $\sigma : X \to X$ a measurable function such that $\#\sigma^{-1}(x) = N$ for all $x \in X$. Finally, let $W : X \to [0, \infty)$ be such that

$$\sum_{y\in X, \sigma(y)=x} W(y) = 1 \tag{2}$$

for almost all $x \in X$. The corresponding Ruelle operator is the linear map $R : L^\infty(X) \to L^\infty(X)$,

$$Rf(x) = \sum_{y\in X, \sigma(y)=x} W(y) f(y).$$

Let $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$ and $\Omega = \prod_1^\infty \mathbb{Z}_N$. For $i_1, \ldots, i_n$ denote the corresponding cylindrical set $\{\omega \in \Omega \mid \omega_1 = i_1, \ldots, \omega_n = i_n\}$ by $A(i_1, \ldots, i_n)$. Finally, let $\tau_i : X \to X$, $i \in \mathbb{Z}_N$, be the right inverses to σ, i.e., $\sigma \circ \tau_i = \mathrm{id}_X$. Then, for each $x \in X$ there exists a unique positive Radon measure P_x on Ω such that

$$P_x(A(i_1, \ldots, i_n)) = W(\tau_{i_1}(x)) W(\tau_{i_2}\tau_{i_1}(x)) \ldots W(\tau_{i_n} \cdots \tau_{i_1}(x)).$$

The book then discusses how the measure P_x shows up in several different places. The connection to wavelet theory is, that things can be arranged so that if φ is a scaling function of an N-scale wavelet, then

$$|\widehat{\varphi}(x + k)|^2 = P_x(\{k\}).$$

The measure is also used to construct harmonic functions, i.e., fixed points for R, $Rh = h$. One such function is simply $h(x) = P_x(\mathbb{N}_0)$.

The author discusses *generalized multiresolution analysis*, GMRA, in chapters seven and eight. This is a family of nested subspaces

$$\cdots \subset V_{j+1} \subset V_j \subset V_0 \subset V_{-1} \subset \cdots \subset V_k \subset V_{k-1} \subset \cdots \subset \mathcal{H} \tag{3}$$

of a Hilbert space $\mathcal{H}$, so that the basic space V_0 is multiple generated, and some other properties, that we will not discuss here, holds. This leads to operator valued measure P_x. In the wavelet theory, this is related to the spectral theory of the representation of $\mathbb{Z}$ on V_0. If (3) corresponds to a multiresolution of $L^2(\mathbb{R})$, then there exists a function $\varphi \in V_0$ such that $\varphi(\cdot + k), \in \mathbb{Z}$, generates V_0. Thus the $\mathbb{Z}$-representation on V_0 is cyclic. This is not the case any more for GMR. Good references are the articles by Baggett and coworkers [1, 2] and the references therein. The spectral decomposition of V_0 is more complicated.

The equation (2) reads now

$$\sum_{i=1}^{N-1} W \circ \tau_i = \mathrm{id}_{\mathcal{H}}$$

and the Ruelle operator acts on $L^\infty(X, \mathcal{H})$.

The Cuntz algebra $\mathcal{O}_N$ is a simple C^*-algebra defined by the relations

$$S_i^* S_j = \delta_{i,j} \mathrm{id}_{\mathcal{H}}, \qquad \sum_{i=0}^{N-1} S_i S_i^* = \mathrm{id}_{\mathcal{H}}. \tag{4}$$

The connections to wavelet theory and GMRA are via the subband-filter functions $m_0, \ldots, m_{N-1}$. The operators S_i are then defined on $L^2([0, 1])$ by

$$S_i f(x) = \sqrt{N} m_i(x) f(Nx).$$

The interplay between the Cuntz algebra and wavelets is described in chapter nine. For that discussion it is also worth taking a look at [3] by Bratteli and Jorgensen.

The book contains not only interesting mathematics. Early on it contains a list of important terms used in mathematics, probability, engineering, and physics, with a short explanation. Each chapters starts with a prelude where the material in that chapter is discussed and brought into context. Finally, each chapter ends with a long section *References and remarks*. Here the reader will find list of interesting references, historical remarks, and explanation of the point of view of the author. As was to be expected, knowing who the author is, the book is full of interesting connections and comments on different parts of mathematics. The book is worth reading for students that are looking for an introduction into wavelets. The long list of exercises will help them cope with the theory. But also a working mathematician might find it interesting to see how iterating function systems and related ideas pop up in different branches of mathematics.

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Positive Polynomials and Sums of Squares. *By Murray Marshall.* Mathematical Surveys and Monographs, Volume 146, AMS, Providence, RI, 2008. 187 pp., hardcover. USD 65, ISBN 978-0-8218-4402-1.

During the last decade three different subjects have shared a common interest in "sums of squares", namely semialgebraic geometry, the moment problem and optimization. The purpose of the book under review is to provide "the beginning student with a short introduction to recent work" in this area as the author writes. According to the reviewer this book is certainly also useful to anybody working in one of the three areas, because it shows unexpected relationships between basic areas of mathematics.

In real algebraic geometry one studies subsets of $\mathbb{R}^n$, which can be described as the set of common zeros of a finite set of polynomials $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[\underline{X}]$, where the last symbol denotes the ring of polynomials in n variables $\underline{X} = (X_1, \ldots, X_n)$ with real coefficients. In semialgebraic geometry one considers sets of the form

$$K = K_S = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \ldots, s\}. \tag{1}$$

(Since the author is working in algebra he makes a distinction between a polynomial $g(\underline{X})$ and the corresponding function $g : \mathbb{R}^n \to \mathbb{R}$. This distinction is not common in analysis.) As an important example let us consider just one polynomial $g_1(\underline{X}) = 1 - X_1^2 - \cdots - X_n^2$. Then the corresponding set K is the unit ball in euclidean n-space. A classical book on semialgebraic geometry is [2].

Given a closed subset K in $\mathbb{R}^n$ (semialgebraic or not), the K-moment problem is to characterize the multisequences $s = (s_\alpha)$ of the form

$$s_\alpha = \int_K x^\alpha \, \mathrm{d}\mu(x), \quad \alpha = (\alpha_1, \ldots \alpha_n) \in \mathbb{N}_0^n, \tag{2}$$

where $N_0 = \{0, 1, \ldots\}$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and μ is a positive measure supported by K. The moment problem can be considered within the framework of harmonic analysis on semigroups, see [1].

To decide if a given polynomial $f \in \mathbb{R}[\underline{X}]$ is a sum of squares of polynomials leads to a semidefinite programming problem, cf. [3]. If a polynomial is a sum of squares of polynomials, it is certainly nonnegative as a function on $\mathbb{R}^n$. A nonnegative polynomial in one variable is a sum of two squares of polynomials, but Hilbert (1888) proved that there exist nonnegative polynomials in two or more variables which cannot be written as a sum of squares. The first concrete example was given by Motzkin in 1967

$$s(X, Y) = 1 + X^2Y^2(X^2 + Y^2 - 3).$$

Nevertheless, a nonnegative polynomial is a sum of squares of rational functions by Artin's solution in 1927 of Hilbert's 17th Problem.

A solution to the K-moment problem was given by Haviland in 1935–36. To formulate it we notice that the dual vector space of $\mathbb{R}[\underline{X}]$ is the set of multisequences (s_α), $\alpha \in \mathbb{N}_0^n$, because a linear functional $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ is uniquely determined by its values $s_\alpha = L(\underline{X}^\alpha)$. Haviland's theorem states that (s_α) is the moment multisequence of a positive measure on K if and only if the corresponding linear functional L is nonnegative on

$$\mathrm{Pos}\,(K) := \{f \in \mathbb{R}[\underline{X}] \mid f(x) \geq 0 \;\forall\, x \in K\}.$$

This result can be viewed as an adaptation of the Riesz representation theorem for positive linear functionals on the vector space of continuous functions with compact support defined on a locally compact space. The fact that a nonnegative polynomial in one variable is a sum of (two) squares then leads to Hamburger's theorem: A sequence (s_n) of real numbers is a moment sequence if and only if

$$\sum s_{i+j} c_i c_j \geq 0$$

for all real sequences (c_n), which are eventually zero. The fact that the set Σ_n of sums of squares of polynomials in n variables is closed in the finest locally convex topology on $\mathbb{R}[\underline{X}]$ and strictly smaller than the set of nonnegative polynomials, when $n \geq 2$, shows that Hamburger's theorem is not valid in dimension higher than 1. Haviland's theorem is not as useful as Hamburger's theorem because it is difficult to test a condition involving all polynomials in Pos (K). This leads to the natural question if there is some representation of the polynomials in this set, if K is given by (1).

Given a finite set S of polynomials as above one introduces

$$T = T_S = \left\{ \sum_{e \in \{0,1\}^s} \sigma_e g_1^{e_1} \cdots g_s^{e_s} \mid \sigma_e \in \Sigma_n \right\},$$

called the preordering of $\mathbb{R}[\underline{X}]$ generated by S. All polynomials in T are certainly nonnegative on the semialgebraic set $K = K_S$.

Theorem 1 (*"Positivstellensatz"*). *For S as above and $f \in \mathbb{R}[\underline{X}]$ the following conditions are equivalent:*

(i) *$f > 0$ on K_S.*
(ii) *There exist $p, q \in T_S$ such that $pf = 1 + q$.*

Stengle proved the result in 1974, but the main ideas were already present in a paper by Krivine from 1964. An important breakthrough came with a theorem of Schmüdgen from 1991 stating that if the semialgebraic set K_S above is assumed to be compact, then any $f \in \mathbb{R}[\underline{X}]$ which is strictly positive on K_S belongs to T_S. The proof was based on another result of his about the K-moment problem.

Theorem 2. *Let $K = K_S$ be a compact semialgebraic set as above and let $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ be a linear functional. Then $L(f) \geq 0$ for all $f \in T_S$ if and only if there exists a positive measure μ on K such that*

$$L(f) = \int_K f(x)\, \mathrm{d}\mu(x), \quad f \in \mathbb{R}[\underline{X}].$$

As an application of Schmüdgen's result we get that any $f \in \mathbb{R}[\underline{X}]$ such that $f(x) > 0$ for all $||x|| \leq 1$ can be represented

$$f(\underline{X}) = \sigma_1(\underline{X}) + (1 - ||\underline{X}||^2)\sigma_2(\underline{X}),$$

with $\sigma_1, \sigma_2 \in \Sigma_n$. All these results and many new refinements can be found in the book under review. An important subject in the presentation is the notion of a preordering in an arbitrary commutative ring A with a unit. Here a preordering is a subset T of A which is stable under sums and products and which contains all squares of elements from A. We have encountered the special preordering T_S in the ring $\mathbb{R}[\underline{X}]$. A treatment of preorderings is also given in the excellent monograph [4].

The book by Marshall presents results from algebra and analysis in a very pleasant way, and it illustrates how important it is to let the two subjects interact.

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Proceedings

Special Functions and Orthogonal Polynomials, *Diego Dominici and Robert S. Maier, eds.* Contemporary Mathematics **471**, the American Mathematical Society, 2008. 218 pp., softcover. USD 69, ISBN 978-0-8218-4650-6.

FROM THE AMS WEBSITE: this volume contains fourteen articles that represent the AMS Special Session on Special Functions and Orthogonal Polynomials, held in Tucson, Arizona in April of 2007. It gives an overview of the modern field of special functions with all major subfields represented, including: applications to algebraic geometry, asymptotic analysis, conformal mapping, differential equations, elliptic functions, fractional calculus, hypergeometric and q-hypergeometric series, nonlinear waves, number theory, symbolic and numerical evaluation of integrals, and theta functions. A few articles are expository, with extensive bibliographies, but all contain original research.

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